Electromagnetic Waves in Toroidal Vessels of Arbitrary Cross Section Filled with Radially Inhomogeneous Dielectric Medium

FERDINAND F. CAP

Abstract —**In this paper, analytical solutions of Maxwell's equations in cylindrical coordinates are presented for toroidal resonators filled with homogeneous or inhomogeneous unmagnetized plasma or another dielectric medium. It is shown that the electromagnetic boundary conditions valid on a conducting toroidal surface of arbitrary meridional cross section can be satisfied by the general solution since the general solution contains an infinite set of arbitrary constants. A method is given to show how these constants and the eigenfrequency of the resonator can be calculated for a given cross section of the toroidal vessel.**

I. Introduction

HEATING OF PLASMAS with electromagnetic waves is an important method to approach ignition in thermonuclear fusion devices. When the wavelength of the electromagnetic wave is small compared to the dimensions of the device, ray-tracing methods of geometrical optics may be used to describe the propagation of the wave in the medium. If, however, the wavelength is of the order of the dimension of the device, wave optical methods should be used to describe the propagation of the waves and to calculate the eigenvalues of a toroidal cavity of arbitrary meridional cross section. Up to now, only approximate methods to treat this problem have been published [1]–[3]. No exact analytical solution is known for a toroidal device filled homogeneously with plasma and no publication for inhomogeneous plasmas is known to the author. In realistic problems, plasmas are anisotropic. For strong confining magnetic fields, the wave field may be neglected in the dielectric tensor. In such a case, Maxwell's equations are still linear, but a little more complicated. Before attacking such a problem it might be useful to explain the new calculation method on a simpler but more unrealistic example. Anisotropic plasmas will be discussed in a forthcoming paper.

Usually wave problems subjected to certain boundary conditions on the confining wall are solved in such a way that the boundary surface is chosen to be a coordinate surface in a coordinate system which is Helmholtz separable, e.g., $r =$ constant for problems with cylindrical wave guides. In this paper, we choose another method. We write down the general solution of Maxwell's equations (containing a set of arbitrary constants). Then we investigate on which surfaces the electromagnetic boundary conditions are satisfied. We will show how to find the arbitrary constants (partial amplitudes) in the general solution in such a manner that these surfaces assume a prescribed cross section.

Manuscript received November 22, 1983; revised April 25, 1984. This work was supported in part by the Austrian Ministry for Science and Research through the Fusion Research Commission of the Austrian Academy of Sciences, Vienna, Austria.

The author is with the Institute for Theoretical Physics, University of Innsbruck, Austria.

II. The Basic Equations for a Homogeneous Medium

In order to expose our method, and as a basis for the calculations for an inhomogeneous plasma, we treat first the homogeneous case. Maxwell's equations, read [4] in cylindrical coordinates, are as follows:

$$-\frac{\partial E_\phi}{\partial z} = -i\omega B_r - \frac{im}{r} E_z \tag{1}$$

$$\frac{1}{r}\frac{\partial r E_\phi}{\partial r} = -i\omega B_z + \frac{im}{r} E_r \tag{2}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -i\omega B_\phi \tag{3}$$

$$-\frac{\partial B_\phi}{\partial z} = \epsilon\epsilon_0\mu_0 i\omega E_r - \frac{im}{r} B_z \tag{4}$$

$$\frac{1}{r}\frac{\partial r B_\phi}{\partial r} = \epsilon\epsilon_0\mu_0 i\omega E_z + \frac{im}{r} B_r \tag{5}$$

$$\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \epsilon\epsilon_0\mu_0 i\omega E_\phi. \tag{6}$$

A dependence $\sim \exp(im\phi)$ and a harmonic time dependence $\sim \exp(i\omega t)$ has been assumed. ϵ is the dielectric constant of the medium. For a cold collisionless unmagnetized plasma, ϵ is constant and given by [5]

$$\epsilon\epsilon_0 = \epsilon_0\left(1 - \omega_P^2/\omega^2\right) \tag{7}$$

where ω is the wave frequency and ω_P is the plasma frequency given by

$$\omega_P^2 = ne^2/m_E\epsilon_0. \tag{8}$$

n is the electron number density and m_E is the electron mass. The equation $\operatorname{div}\vec{B} = 0$ is automatically satisfied by the solution of (1)–(3), and $\operatorname{div}(\epsilon\epsilon_0\vec{E})$ yields the deviation from quasineutrality. If the plasma is inhomogeneous, the plasma density n in (8) becomes a function of space.

0018-9480/84/1000-1336$01.00 ©1984 IEEE

III. The General Solution and the Boundary Conditions

The general solution of Maxwell's equations in cylindrical coordinates is well known [6]. For $\partial/\partial\phi = 0$ and $\epsilon = \text{constant}$, it can be written in the form

$$E_\phi = \sum_{k=0} A_k Z_1\left(\sqrt{\gamma^2 - k^2}\, r\right)\cos kz + \sum_{k_1} B_{k_1} Z_1\left(\sqrt{\gamma^2 - k_1^2}\, r\right)\sin k_1 z \tag{9}$$

$$B_\phi = \sum_{k_2=0} \bar{A}_{k_2} Z_1\left(\sqrt{\gamma^2 - k_2^2}\, r\right)\cos k_2 z + \sum_{k_3} \bar{B}_{k_3} Z_1\left(\sqrt{\gamma^2 - k_3^2}\, r\right)\sin k_3 z \tag{10}$$

$$B_r = -\frac{i}{\omega r}\frac{\partial E_\phi r}{\partial z} = \frac{i}{\omega}\left\{\sum_k k A_k Z_1 \cdot\left(\sqrt{\gamma^2 - k^2}\, r\right)\sin kz - \sum_{k_1} k_1 B_{k_1} Z_1\left(\sqrt{\gamma^2 - k_1^2}\, r\right)\cos k_1 z\right\} \tag{11}$$

$$B_z = \frac{i}{\omega r}\frac{\partial E_\phi r}{\partial r} = \frac{i}{\omega}\left\{\sum_k A_k \sqrt{\gamma^2 - k^2}\, Z_0 \cdot\left(\sqrt{\gamma^2 - k^2}\, r\right)\cos kz + \sum_{k_1} B_{k_1}\sqrt{\gamma^2 - k_1^2}\, Z_0\left(\sqrt{\gamma^2 - k_1^2}\, r\right)\sin k_1 z\right\} \tag{12}$$

$$E_r = \frac{i\omega}{\gamma^2 r}\frac{\partial B_\phi r}{\partial z} = \frac{i\omega}{\gamma^2}\left\{-\sum_{k_2} \bar{A}_{k_2} k_2 Z_1 \cdot\left(\sqrt{\gamma^2 - k_2^2 r}\right)\sin k_2 z + \sum_{k_3} k_3 \bar{B}_{k_3} Z_1\left(\sqrt{\gamma^2 - k_3^2}\, r\right)\cos k_3 z\right\} \tag{13}$$

$$E_z = -\frac{i\omega}{\gamma^2 r}\frac{\partial B_\phi r}{\partial r} = -\frac{i\omega}{\gamma^2}\left\{\sum_{k_2=0} \bar{A}_{k_2}\sqrt{\gamma^2 - k_2^2}\, Z_0 \cdot\left(\sqrt{\gamma^2 - k_2^2}\, r\right)\cos k_2 z + \sum_{k_3} \bar{B}_{k_3}\sqrt{\gamma^2 - k_3^2}\, Z_0\left(\sqrt{\gamma^2 - k_3^2}\, r\right)\sin k_3 z\right\} \tag{14}$$

where

$$\gamma = \sqrt{\epsilon\epsilon_0\mu_0}\,\omega \tag{15}$$

and ω is the eigenfrequency of the toroidal cavity. k, k_1, k_2, and k_3 are separation constants, depending on the boundary conditions.

Equations (9)–(14) satisfy (1)–(6) and $\operatorname{div}\vec{B} = 0$, $\operatorname{div}\vec{E} = 0$. Z_0 and Z_1 are appropriate cylinder functions, i.e., a superposition of Bessel's function of the first kind J and of the second kind Y (Neumann function). The $\bar{A}_k$, A_k, $\bar{B}_k$, B_k are constants determined by the boundary conditions and by the given form of the cross section of the toroidal vessel.

From (9)–(14) we see that for $\partial/\partial\phi = 0$ two special solutions exist. Since propagation takes place in the ϕ direction (and *not* in the z direction as in the usual situation with a cylindrical waveguide), the solution E_r, E_z, B_ϕ may be called a transverse electric (TE) wave and B_r, B_z, E_ϕ may be termed a transverse magnetic (TM) wave. Since the boundary conditions are different for the TE and TM waves, the eigenvalues ω (or γ) will, *in general*, but not always, be different. If the k values are different, TE and TM waves of the same eigenfrequency ω may satisfy the boundary conditions on an arbitrary cross section of the conductor.

The electromagnetic boundary conditions demand continuity of the tangential component E_t of the electric and of the normal component B_n of the magnetic field on each interface. On the surface of a perfect conductor both components vanish. If the cross section in the meridional plane $\phi = \text{constant}$ of the interface or the surface is described by $z(r)$ in cylindrical coordinates (see Fig. 1), the tangential and normal components are obtained by projections of the radial and z components. We consider first the magnetic field (see Fig. 2)

$$B_t = B_r\cos\alpha + B_z\sin\alpha \tag{16}$$

$$B_n = -B_r\sin\alpha + B_z\cos\alpha \tag{17}$$

where

$$\alpha = \operatorname{arctg}\left(\frac{dz}{dr}\right). \tag{18}$$

Now magnetic field lines are lines on which the magnetic field is purely tangential. Therefore, $B_n = 0$ along magnetic field lines. The condition $B_n = 0$ describes also a perfect conductor. Thus, special magnetic field lines (i.e., those along which E_t vanishes) describe the conductor $z = z(r)$ and determine the meridional cross section of the vessel. Magnetic field lines are described by $B_n = 0$ or by the differential equation

$$\frac{dr}{B_r} = \frac{dz}{B_z} \tag{19}$$

which follows from (17) and (18). Inserting B_z and B_r from (11) and (12), we may integrate to obtain $z(r)$ in the form

$$rE_\phi = \text{constant} = Q. \tag{20}$$

These curves describe the projection of the magnetic field lines on a meridional plane (and $Q = 0$ describes the meridional cross-section surface of the conductor $z(r)$ since E_ϕ must vanish on a conductor situated as in Fig. 1).

Equations (19) and (20) describe the TM wave given by E_ϕ, B_r, B_z. The electric boundary condition $E_t = 0$ is satisfied, since $E_r = E_z = 0$ and $E_\phi = 0$ on the conductor.

Electric field lines are described by $E_n = 0$ and a metallic wall is described by $E_t = 0$. If $z = \bar{z}(r)$ is the expression for the meridional projection of the electric field lines and

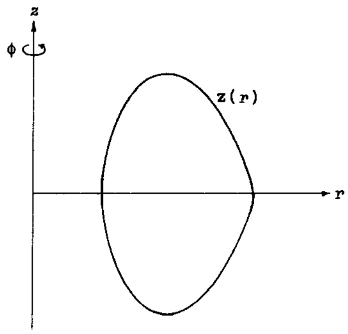


Fig. 1. Cylindrical coordinates.

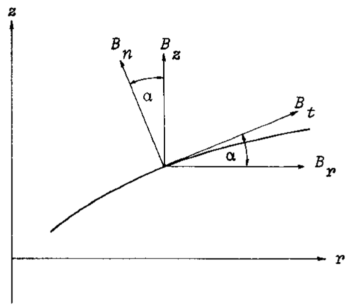


Fig. 2. Projections of the field components.

$\beta = \bar{z}'$, then

$$E_t = E_r \cos\beta + E_z \sin\beta \tag{21}$$

$$E_n = -E_r \sin\beta + E_z \cos\beta \tag{22}$$

and from $E_n = 0$ and (13) and (14) we obtain

$$rB_\phi = \text{constant} = D \tag{23}$$

for the projection of the electric field lines $\bar{z}(r)$. These field lines describe the TE wave given by B_ϕ, E_r, E_z. $E_t = 0$ or $E_r d\bar{z} - E_z\, dr = 0$ yields after inserting from (13) and (14)

$$\frac{\partial B_\phi r}{\partial r} + \frac{\partial B_\phi r}{\partial z}\frac{d\bar{z}}{dr} = 0. \tag{24}$$

The orthogonal trajectories $u(r)$ to these curves $\bar{z}(r)$ are defined by

$$\frac{d\bar{z}}{dr} = -1 \Big/ \frac{du}{dr}. \tag{25}$$

Thus, (24) may be written $(\partial B_\phi r/\partial r)\cdot(dz/dr) - \partial B_\phi r/\partial z = 0$ and $u \equiv z$. That means that the condition $E_t = 0$ yields curves orthogonal to the electric field lines and parallel to the conductor. The magnetic boundary condition $B_n = 0$ on the conductor is satisfied by the TE wave, since then $B_r = B_z = 0$. In order to find these magnetic fields lines $z(r)$ along which E_t vanishes, we use (19) in the form

$$\frac{dz}{dr} = \frac{B_z}{B_r}. \tag{26}$$

Along $z(r)$, and for vanishing E_t, (21) may be written $dz/dr = -E_r/E_z$. Combining this with (26) we obtain

$$E_r B_r + E_z B_z = 0 \tag{27}$$

which is equivalent to $\vec{E}\cdot\vec{B} = 0$ or $\vec{E} \perp \vec{B}$, since $E_\phi = 0$ on the conducting wall. Equations (20) or (27) allow us to find the surfaces (meridional curves $z(r)$) on which $E_t = 0$, $B_n = 0$ is valid.

If the TE and TM waves are assumed to have the *same* frequency, they must have a different k in order to be able to satisfy the boundary conditions on a conductor of arbitrary cross section (see Section IV).

IV. Boundary Conditions on an Arbitrary Cross Section

In order to obtain a smooth analytical boundary curve in the frame given by expression (20), a trial and error method in choosing the value k and the points r_i, $z_i = z(r_i)$ is necessary. According to our experience, about four points, r_i, $z_i = z(r_i)$, $i = 1, \cdots, 4$, represent the minimum to define a cross section $z = z(r)$ roughly. The more points one takes into account, the more partial waves have to be taken into consideration and the more the cross section may be varied. For $i = 1, \ldots, N$, $Q = 0$, (9) and (20) yield the N homogeneous linear equations for the A_k, B_{k_1}

$$E_\phi \equiv \sum_k A_k Z_1\left(\sqrt{\gamma^2 - k^2}\, r_i\right)\cos kz_i + \sum_{k_1} B_{k_1} Z_1\left(\sqrt{\gamma^2 - k^2}\, r_i\right)\sin k_1 z_i = 0. \tag{28}$$

If we take p partial waves into account, i.e., p different values of k and k_1, we have altogether $4p = N$ unknown coefficients, since each $A_k Z_1$, $B_{k_1} Z_1$ contains two constants. Since linear homogeneous equations can be solved only if the determinant of the coefficients vanishes, we calculate the eigenvalue γ from the vanishing of the determinant (for given k's). For $B_{k_1} = 0$, $2p = N$, $p = 2$, the procedure is very simple. From (28) we obtain

$$a_0 J_1(\gamma r_i) + f_0 Y_1(\gamma r_i) + \left[a_1 J_1\left(\sqrt{\gamma^2 - k^2}\, r_i\right) + b_1 Y_1\left(\sqrt{\gamma^2 - k^2}\, r_i\right)\right]\cos kz_i = 0 \tag{29}$$

where $i = 1, \cdots, 4$, a_0, f_0, a_1 are other designations for the constants and $b_1 = 1$. For $k = 7.4$ and $r_1 = 0.6$, $z_1 = 0$, $r_2 = 1.3$, $z_2 = 0$, $r_3 = 0.83$, $z_3 = 0.28$, $r_4 = 1.10$, $z_4 = 0.20$ (all values in meters) we obtained $\gamma = 8.17629964697$, $\omega/2\pi \approx 500$ MHz for $n = 1.23 \times 10^{15}$ pm^{-3}, and the cross section $z(r)$ of the conductor shown in Fig. 3. The meridional cut of the conductor and the projection of the magnetic fieldlines of the TM wave into the meridional plane may be represented together by

$$z(r) = \frac{1}{k}\arccos\left[\frac{Q - r\left(a_0 J_1(\gamma r) + f_0 Y_1(\gamma r)\right)}{r\left(a_1 J_1\left(\sqrt{\gamma^2 - k^2}\, r\right) + Y_1\left(\sqrt{\gamma^2 - k^2}\, r\right)\right)}\right] \tag{30}$$

where we found $a_0 = 0.173255$, $f_0 = -0.779360$, $a_1 = 0.445031$. For the r_i, z_i chosen above, the denominator of (30) has no zeroes and the expression in the brackets lies in the interval $-1 \leqslant [\;] \leqslant 1$. We now consider the TE wave. Assuming the electric field symmetric in z, $(B_k = \overline{B}_k = 0)$,

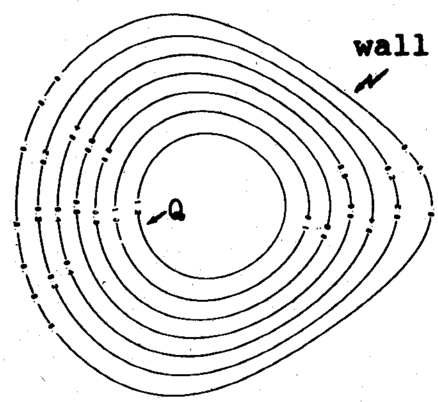


Fig. 3. TM wave and cross section of torus for $\gamma = 8.1763$.

we need $p = N/2$ particular solutions. Then (10) reads

$$B_\phi = \sum_{s=0}^{p-1} \left[\bar{a}_s J_1\left(\sqrt{\gamma^2 - k_s^2}\, r\right) + \bar{b}_s Y_1\left(\sqrt{\gamma^2 - k_s^2}\, r\right)\right] \cos k_s z \tag{31}$$

and E_r and E_z may be calculated from (13) and (14). Inserting for E_r, E_z as well as for B_r and B_z—calculated from (11), (12), and (28)—the boundary condition (27) yields $N = 2p$ linear equations for the $\bar{a}_s, \bar{b}_s$, if $r = r_i$, $z = z_i$, $i = 1, \cdots, N$ and the $k_s < \gamma$ are given. For example, by dividing the possible interval $0 \leqslant k_s \leqslant \gamma$ roughly into equal steps, we may choose $p = 10$ and $k_0 = 0$; $k_1 = 4.5$; $k_2 = 5$; $k_3 = 5.5$; $k_4 = 6$; $k_5 = 6.5$; $k_6 = 7.2$; $k_7 = 7.4$; $k_8 = 7.8$; $k_9 = 8$. This yields the electric field lines (23) together with magnetic field lines (20) of the TE and TM hybrid mode as shown in [11].

V. Inhomogeneous Medium

In an inhomogeneous plasma, ϵ and n are no longer constant. Since (1)–(6) are valid also for an inhomogeneous plasma, we derive an equation for E_ϕ from (1), (2), and (6). The result is (for $m = 0$)

$$\frac{\partial^2 E_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial E_\phi}{\partial r} - \frac{1}{r^2} E_\phi + \frac{\partial^2 E_\phi}{\partial z^2} + \omega^2 \epsilon \epsilon_0 \mu_0 E_\phi = 0. \tag{32}$$

For $\epsilon = \text{constant}$, $\omega^2 \epsilon \epsilon_0 \mu_0 = \gamma^2$, and a z dependence $\exp(ikz)$, this equation is solved by (9).

We now assume that the electric permeability ϵ is a function depending only on r ("radially inhomogeneous" plasma) and that

$$E_\phi = E_\phi^{(0)}(r) \cos kz + E_\phi^{(1)}(r) \sin k_1 z. \tag{33}$$

We thus have $(i = 0, 1)$

$$E_\phi^{i\prime\prime} + \frac{1}{r} E_\phi^{i\prime} - \frac{1}{r^2} E_\phi^i + \left(\gamma^2 - k_i^2\right) E_\phi^i + \omega^2 \epsilon(r) \epsilon_0 \mu_0 E_\phi^i = 0. \tag{34}$$

Fusion devices have density profiles like [7]

$$n(r) = n_1 - n_0 (r - R)^2 \tag{35}$$

where $R \simeq 0.95$ m and $a_0 \simeq 0.35$ m (major and minor torus radius). Furthermore, we assume $n_1 = 1.225 \times 10^{15}$ particles m^{-3} and $n_0 = 10^{16}$. From (7), (8), and (33) we then have

$$\omega^2 \mu_0 \epsilon_0 \epsilon(r) = \gamma^2 + (a + br + cr^2)$$

$$\gamma^2 = \omega^2 \epsilon_0 \mu_0 - \frac{e^2 \mu_0 n_1}{m_E}. \tag{36}$$

Here the parameters a, b, c are defined by

$$a = \frac{e^2 n_0 R^2 \mu_0}{m_E} = 319.5 \tag{37}$$

$$b = -2 \frac{e^2 n_0 \mu_0}{m_E} R = -672.7 \tag{38}$$

$$c = \frac{e^2 n_0 \mu_0}{m_E} n = 354.1. \tag{39}$$

Then (34) becomes $(i = 0, 1)$

$$E_\phi^{i\prime\prime} + \frac{1}{r} E_\phi^{i\prime} - \frac{1}{r^2} E_\phi^i + \left(\gamma^2 + a - k_i^2\right) E_\phi^i + (br + cr^2) E_\phi^i = 0. \tag{40}$$

For $b = 0$, (40) is a Bessel wave equation [8]; for $b \neq 0$, we have a Bôcher equation [9]. No table was available for Bôcher functions or for their zeroes. Thus, a numerical integration was necessary. In order to satisfy (20) in the four points r_i $(i = 1, \cdots, 4)$ determining the meridional cross section $z(r)$, we have to solve (40) numerically in such a way that the TM mode satisfies

$$E_\phi^0(r) \cos kz + E_\phi^1(r) \sin k_1 z = 0 \tag{41}$$

for the r_i, z_i. These four conditions yield the eigenvalue γ. In the numerical integration of (40), a first approximation for γ may be taken from Section IV (homogeneous medium) and may be improved by a *regula falsi* procedure (see later). In the analytical expression (30), the Bessel functions have to be replaced by the Bessel wave functions or Bôcher functions defined by (40). Then (19) yields again (20) so that the magnetic field lines are given again by $rE_\phi = Q$. On the metallic surface, (27) must be satisfied too. We thus need not only B_r and B_z but also E_r and E_z. In order to find E_r and E_z, we need $B_\phi(r, z)$. We thus have to integrate the two equations $(i = 2, 3)$

$$-\frac{k_i^2}{\epsilon} B_\phi^{(i)} + \frac{d}{dr}\left(\frac{1}{\epsilon r} \frac{dr B_\phi^{(i)}}{dr}\right) + \epsilon_0 \mu_0 \omega^2 B_\phi^{(i)} = 0 \tag{42}$$

for the TE wave.

This equation can be derived from (3), (4), and (5). Using (33), (35), (36), and (37)–(39), as well as $i = 2$, $B_\phi^{(3)} = 0$, $B_\phi^{(1)} = u(r)$, we obtain

$$u'' + \frac{1}{r} u' - \frac{1}{r^2} u + (\gamma^2 - k^2) u + (a + br + cr^2) u - \frac{b + 2cr}{\gamma^2 + a + br + cr^2}\left(\frac{1}{r} u + u'\right) = 0. \tag{43}$$

The projection of the electric field lines onto the meridional plane is given by (23), where B_ϕ may be given by

$$B_\phi(r, z) = u_1(r) + u_2(r) \cos kz \tag{44}$$

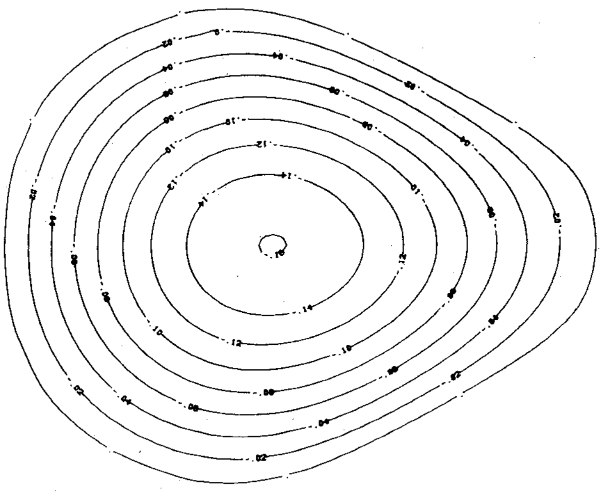


Fig. 4. Magnetic field for inhomogeneous medium.

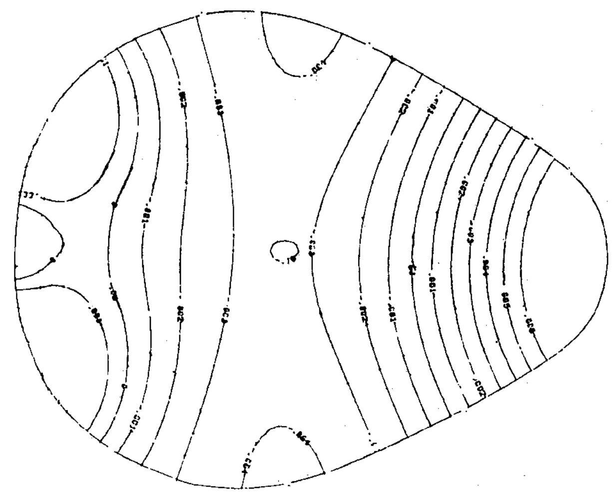


Fig. 5. Electric field for inhomogeneous medium.

(for two solutions $k=0$ and $k\neq 0$ symmetric in z). Since (43) is a homogeneous differential equation of second order, its eigenvalue γ depends on the boundary conditions only and not on the initial conditions assumed for the numerical integration. If we designate by $u_1^{(1)}, u_1^{(2)}$ the solutions for $k=0$ by $u_2^{(1)}, u_2^{(2)}$ the solutions for $k\neq 0$ (for example $k=7.4$) for arbitrary but different initial conditions $u_1^{(1)}(0), u_1'^{(1)}(0)$; $u_1^{(2)}(0), u_2'^{(2)}(0)$, etc., then (44) reads

$$B_\phi(r,z)=\bar{a}_0 u_1^{(1)}(\gamma r)+\bar{f}_0 u_1^{(2)}(\gamma r) + \left[\bar{a}_1 u_2^{(1)}\left(\sqrt{\gamma^2-k^2}\,r\right)+\bar{b}_1 u_2^{(2)}\left(\sqrt{\gamma^2-k^2}\,r\right)\right]\cos kz. \quad (45)$$

Inserting this into the boundary condition (24) we obtain for the four points r_i, z_i $(i=1,\cdots,4)$ four homogeneous linear equations for $\bar{a}_0, \bar{f}_0, \bar{a}_1, \bar{b}_1$. For the first guess of γ, the determinant $D(\gamma)$ of the coefficients will not vanish, since in order to be able to execute the numerical integration of (43) we had to assume an arbitrary approximate value for γ $(\gamma>k)$, for example the value of the TM wave given by (41). A *regula falsi* procedure applied on $D(\gamma)\to 0$ will improve the value of γ and a renewed numerical integration of (43) will deliver a better approximation for γ. The process converges rapidly. When applied on (40) for the TM wave for

$$\begin{array}{llll} r_1=0.605 & z_1=0.065 & r_2=1.295 & z_2=0.037 \\ r_3=0.83 & z_3=0.28 & r_4=1.10 & z_4=0.20 \\ k_0=0 & k_1=7.4 & & \\ a=319.5 & b=-672.7 & c=354.1 & \end{array} \quad (46)$$

and

$$E_\phi^0=a_0 y_1^{(1)}(\gamma r)+f_0 y_1^{(2)}(\gamma r)$$
$$E_\phi^1=a_1 y_2^{(1)}\left(\sqrt{\gamma^2-k_1^2}\,r\right)+b_1 y_2^{(2)}\left(\sqrt{\gamma^2-k_1^2}\,r\right) \quad (47)$$

with the arbitrary initial conditions

$$\begin{array}{llll} y_1^{(1)}(0)=1.5 & y_1'^{(1)}(0)=1 & y_1^{(2)}(0)=1 & y_1'^{(2)}(0)=1 \\ y_2^{(1)}(0)=1.5 & y_2'^{(1)}(0)=1 & y_2^{(2)}(0)=1 & y_2'^{(2)}(0)=1 \end{array} \quad (48)$$

we obtained after the eleventh integration $\gamma=7.8246508621$, $a_0=1$, $f_0=-1.4543$, $a_1=1.2624$, $b_1=-1.9393$. The magnetic field lines are shown in Fig. 4. For the TE wave we may either assume the same γ and calculate one of the k_i as eigenvalue or we may assume all k_i and calculate the eigenvalue γ. A procedure based on (43), (44), (45), and (24) yields for the same $\gamma=7.8246508621$ for particular solutions ($k_1=0$, k_2 to be determined) the result $\bar{a}_0=1$, $\bar{f}_0=-1.3531$, $\bar{a}_1=-0.71067$, $\bar{b}_1=1.0167$, and $k_2=7.2580671810$. The electric field lines are shown in Fig. 5.

VI. The Nonaxisymmetric Case

When $m\neq 0$, the situation is a little more complicated, but it can be handled. Combining (1), (5), (2), and (4) in a suitable way, we obtain

$$E_z=\frac{i}{M}\left(m\frac{\partial rE_\phi}{\partial z}-\omega r\frac{\partial rB_\phi}{\partial r}\right) \quad (49)$$

$$E_r=\frac{i}{M}\left(m\frac{\partial rE_\phi}{\partial r}+\omega r\frac{\partial rB_\phi}{\partial z}\right) \quad (50)$$

$$B_r=\frac{i}{M}\left(m\frac{\partial rB_\phi}{\partial r}-r\epsilon(r)\epsilon_0\mu_0\omega\frac{\partial rE_\phi}{\partial z}\right) \quad (51)$$

$$B_z=\frac{i}{M}\left(m\frac{\partial rB_\phi}{\partial z}+r\epsilon(r)\epsilon_0\mu_0\omega\frac{\partial rE_\phi}{\partial r}\right) \quad (52)$$

where $M=\epsilon(r)\epsilon_0\mu_0\omega^2r^2-m^2$. Equations (49)–(52) satisfy (1), (2), (4), and (5). We see that we have again two waves: a TE wave ($E_\phi=0$, $B_\phi\neq 0$, with the components E_z, E_r, and B_r, B_z) and a TM wave ($B_\phi=0$, $E_\phi\neq 0$ with B_z, B_r, and E_z, E_r). In order to solve (49)–(52) together with (3) and (6), we make the ansatz

$$\begin{array}{ll} E_r(r,z)=\bar{E}_r(r)\sin kz & B_r(r,z)=\bar{B}_r(r)\cos kz \\ E_\phi(r,z)=\bar{E}_\phi(r)\sin kz & B_\phi(r,z)=\bar{B}_\phi(r)\cos kz \\ E_z(r,z)=\bar{E}_z(r)\cos kz & B_z(r,z)=\bar{B}_z(r)\sin kz. \end{array} \quad (53)$$

Inserting into (49)–(52) and (3) and (6), we obtain

$$\overline{B}_\phi'' + \frac{1}{r}\overline{B}_\phi' + \left(\epsilon(r)\epsilon_0\mu_0\omega^2 - k^2\right)\overline{B}_\phi + \frac{1}{r^2}\overline{B}_\phi - \frac{m^2}{r^2}\overline{B}_\phi - \frac{2\overline{B}_\phi' m^2}{r\left(\epsilon(r)\epsilon_0\mu_0\omega^2 r^2 - m^2\right)} - \frac{2\overline{B}_\phi\epsilon(r)\epsilon_0\mu_0\omega^2}{\epsilon(r)\epsilon_0\mu_0\omega^2 r^2 - m^2} + \frac{2mk\overline{E}_\phi\epsilon(r)\epsilon_0\mu_0\omega^2}{\epsilon(r)\epsilon_0\mu_0\omega^2 r^2 - m^2} + \mu_0\epsilon_0\epsilon'(r) \cdot \left(rmk\omega\overline{E}_\phi - \omega^2\overline{B}_\phi r - \omega^2 r^2\overline{B}_\phi'\right) = 0 \qquad (54)$$

where $\epsilon(r)$ is given by

$$\epsilon(r) = \frac{1}{\omega^2\mu_0\epsilon_0}\left(\gamma^2 + a + br + cr^2\right) \qquad (55)$$

and γ is defined by (36). For the homogeneous medium ($\epsilon' = 0$), we receive an equation obtained earlier [12]. Due to the inhomogeneity, the analytic method used there can no longer be applied. Thus, (54) and the corresponding equation for $\overline{E}_\phi$ have to be solved using a computer. The method of solution will be the same as described in Section V of this paper and in [12]. However, the form of the waveguide wall on which the boundary conditions are to be satisfied, will be helical again [12].

VII. Acknowledgment

The author is indebted to MM Tranquillini and Stoessel for help in preparing computer codes.

References

[1] M. Brambilla and V. Finzi, "Electromagnetic eigenmodes of the toroidal cavity," *IEEE Trans. Plasma Sci.*, vol. PS-2, pp. 112–114, Sept. 1974.

[2] F. Cap and R. Deutsch, "Toroidal resonators for electromagnetic waves," *IEEE Trans. Microwave Theory*, vol. MTT-26, pp. 478–486, July 1978.

[3] F. Cap and R. Deutsch, "Toroidal resonators for electromagnetic waves II," *IEEE Trans. Microwave Theory*, vol. MTT-28, pp. 700–703, July 1980.

[4] J. Jackson, *Classical Electrodynamics*. New York: Wiley 1975.

[5] F. Cap, *Handbook on Plasma Instabilities*, New York: Academic: vol. 1, p. 186, 1976.

[6] *American Institute of Physics Handbook*. New York: McGraw-Hill, various editions.

[7] "Plasma physics and controlled nuclear fusion research 1978," in *7th IAEA Conf. Proc.*, (Innsbruck, Austria), vol. 1, Aug. 23–30, 1978, pp. 21, 117, 125, 138.

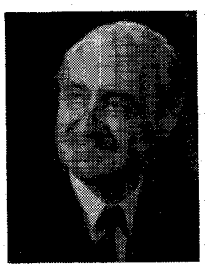
[8] P. Moon and D. Spencer, *Field Theory Handbook*. Berlin: Springer, pp. 155, 171, 1971.

[9] ——, *Field Theory Handbook*. Berlin: Springer, p. 145.

[10] F. Cap, "Oscillations of membranes of nearly arbitrary form," *Acta Phys. Austriaca*, vol. 54, pp. 69–74, 1982.

[11] F. Cap, *Handbook on Plasma Instabilities*. New York: Academic Press, vol. 3, p. 1240, 1982.

[12] F. Cap, "Toroidal resonators and waveguides of arbitrary cross section," *IEEE Trans. Microwave Theory*, vol. MTT-29, pp. 1053–1059, Oct. 1981.



Ferdinand F. Cap was born in 1924 near Vienna, Austria. He studied physics, mathematics, and chemistry from 1942–1946 at the University of Vienna, Vienna, Austria.

He became a Lecturer at the University of Innsbruck, Austria, in 1949 after a stay at the Zürich Technical University. He was promoted to Assistant Professor in 1955, Associate Professor with tenure in 1957, and Full Professor in 1960. In 1967, he was Guest Professor for plasma physics in New York. In 1971, he became a Senior Research Associate of NASA at Goddard Space Center and in 1979 he was a visitor at the Princeton Plasma Physics Laboratory. For 10 years he was Austria's Scientific Representative at the UN in New York. He has published more than 100 papers in physics and several textbooks about nuclear reactors and plasma physics.